Indiana State Mathematics Contest 2016

Geometry

Do not open this test booklet until you have been advised to do so by the test proctor.

This test was prepared by faculty at **Indiana University Purdue University Columbus**

Next year's math contest date: Saturday, April 22, 2017

1) What is the midpoint of the line segment connecting the points $(1,1)$ and $(2,-2)$?
   a) $(1.5, 1.5)$
   b) $(1.5, -0.5)$
   c) $(1,2)$
   d) $(0,0)$
   e) None of the above.

2) A farmer has a rectangular plot that is 4 ft wide by 6 ft long. What is the area of the plot?
   a) 24 in^2
   b) 48 yd^2
   c) 10 ft^2
   d) 3456 in^2
   e) None of the above.

3) An ellipse in the *xy*-plane centered at the origin is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a, b > 0$. You are told that the quantity e is defined to be $e = \sqrt{1 - \frac{b^2}{a^2}}$. Further, you are told that $e = 0$. What can you conclude about the ellipse?
   a) The ellipse is an infinite line segment along the major axis.
   b) The ellipse is also a parabola.
   c) The ellipse is a circle of radius 1.
   d) The ellipse is a circle of radius a.
   e) None of the above.

4) Which of the following statements is true?
   a) A line intersects a circle exactly one time.
   b) A triangle has 4 vertices.
   c) The largest angle in a triangle must be no more than 90 degrees.
   d) The interior angles of a 15 sided polygon sum to 2340 degrees.
   e) None of the above.

5) Identify the point on the line $y = 2x$ that has the shortest distance to the point $(0,1)$.
   a) $(0,0)$
   b) $(0,1)$
   c) $(0.4,0.8)$
   d) $(0.5,1)$
   e) None of the above.

6) Which of the following equations describes all of the points in the *xy*-plane that are exactly 3 units from the point $(0,0)$?
   a) $x^2 + y^2 = 9$
   b) $(x-3)^2 + (y-3)^2 = 1$
   c) $x^2 + y^2 = 3$
   d) $y = 3x$
   e) None of the above.

7) Visualize in 3D *xyz*-space the points
   A: where $x = 1, y = 0$, and $z = 0$; and
   B: where $x = 1, y = 1$, and $z = 0$.
   Now suppose the line segment connecting *A* and *B* is revolved about the *y*-axis. What is the surface area of the resulting surface of revolution?
   a) 2π
   b) 4π
   c) π
   d) 2
   e) None of the above.

8) Suppose that an equilateral triangle has a positive perimeter of *L*. Calculate the area of the triangle.
   a) $\frac{L^2}{12\sqrt{3}}$
   b) $\frac{L^2}{9}$
   c) $\frac{L^2}{2\sqrt{3}}$
   d) $\frac{L}{2\sqrt{3}}$
   e) None of the above

9) Suppose you have two cubes (one smaller and one larger) and you are told that the space diagonal (not a face diagonal) of the larger cube is 14 times the length of the smaller cube's space diagonal. The volume of the larger cube is a multiple of the volume of the smaller cube. Determine that multiple.
   a) 14
   b) $\sqrt{3}$
   c) $14\sqrt{3}$
   d) 2744
   e) None of the above.

10) Ignoring units, you are told that the area of a circle is equal to its circumference. What is the area of that circle? You may assume the circle is not a point.
   a) 2
   b) 4π
   c) 4
   d) $\frac{\pi^2}{4}$
   e) None of the above.

11) Which of the following is NOT possible for a convex polyhedron?
   a) Has exactly 4 vertices, 6 edges, and 4 faces.
   b) Has exactly 3 vertices, 4 edges, and 4 faces.
   c) Has exactly 8 vertices, 12 edges, and 6 faces.
   d) Has exactly 51 vertices, 100 edges, and 51 faces.
   e) None of the above.

12) Consider in 3D *xyz*-space the *xy*-plane and the point $(1,3,0)$ ($x = 1, y = 3,$ and $z = 0$). Identify the point or points in the *xy*-plane that are closest to (1,3,0).
   a) (1,2,0)
   b) (1,1,1)
   c) (1,3,0)
   d) (0,3,1)
   e) None of the above.

13) Find the equation of the line that is perpendicular to the line segment connecting (1,1) to (2,2) and goes through the point (2,1).
   a) $x = 2$
   b) $y = 1$
   c) $x + y = 3$
   d) $y = -x + 2$
   e) None of the above.

14) Suppose that in a square with side length 2, there are regions of two colors: red and blue. There are *N* disjoint red pieces all with different areas. The largest red piece has an area of 0.5 and the next largest piece has an area that is exactly $\frac{3}{4}$ of the largest piece. This rule is true for all red pieces: any red piece (other than the largest) has an area that is exactly $\frac{3}{4}$ of the piece directly above it in size. What is the minimum value for *N* that guarantees that the red pieces comprise at least $\frac{1}{4}$ of the area of the entire square?
   a) 2
   b) 3
   c) 10
   d) 15
   e) None of the above.

15) Suppose that *a* and *b* are positive constants and that the points (a, b), $(3a, b)$, and $(3a, 3b)$ are the three vertices of a triangle. What is the length of the longest side of the triangle?
   a) $2a + 2b$
   b) $\sqrt{a^2 + b^2}$
   c) $2\sqrt{a^2 + b^2}$
   d) $4a^2 - 4b^2$
   e) None of the above.

16) Below in Figure 1, we see a square that has been cut resulting in 3 triangles and 1 rectangle. The areas of the triangles are labeled. What is the area of the rectangle?
   a) 2
   b) 4
   c) 8
   d) 22
   e) None of the above.

17) Below in Figure 2, we see an annulus formed between two concentric circles (as shown in the shaded region). If the radius of the smaller circle is 2, what should the radius of the larger circle be if the annulus is to have an area 4 times that of the smaller circle?
   a) 20
   b) $2\sqrt{5}$
   c) 8
   d) 6
   e) None of the above.

18) Below, which describes an equation for a line parallel to the line segment connecting (3,4) and (4,6)?
   a) $x + 2y = 1$
   b) $y = \frac{1}{2}x$
   c) $y = 2x - 2$
   d) $-y = 2x + 1$
   e) None of the above.

19) Referring to Figure 3, find $x = |CD|$. Congruent angles and right angles are marked.
   a) $\frac{2}{3}$
   b) $\sqrt{13}$
   c) $\frac{26}{5}$
   d) $\sqrt{5}$
   e) None of the above.

Figure 1:

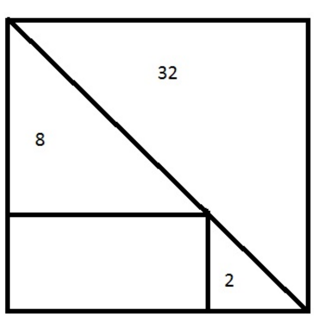


Figure 2:

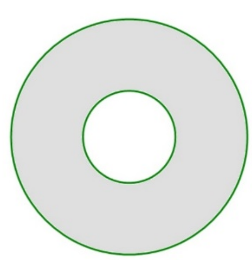
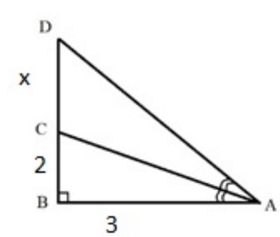


Figure 3:



20) Suppose that for $0^o < \theta < 90^o$, we define the function $f(\theta) = (\cos(\theta))^{41}(\sin(\theta) - \cos(\theta))$. Compute $f(45^o)$.

a) 0
b) $2^{-\frac{41}{2}}$
c) $\frac{\sqrt{2}}{2}$
d) -0.6
e) None of the above.

21) Calculate the length of a chord that is a perpendicular bisector of a radius of length 6.

a) $6\sqrt{3}$
b) $\sqrt{3}$
c) $\sqrt{6}$
d) 2
e) None of the above.

22) Referring to Figure 4, what is the measure of angle *A* in radians?

a) $\frac{\pi}{3}$
b) $\frac{\pi}{6}$
c) $\frac{5\pi}{6}$
d) π
e) None of the above.

23) For the triangle in Figure 5, what is the length of the missing side?

a) $\sqrt{17 - \frac{8}{\sqrt{2}}}$
b) 2
c) $4 - \frac{1}{\sqrt{2}}$
d) 3
e) None of the above.

24) What is the area of the triangle in Figure 6?

a) $\frac{3\sqrt{3}}{2}$
b) 3
c) $6\sqrt{2}$
d) $3\sqrt{2}$
e) None of the above.

Figure 4:

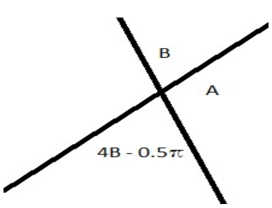


Figure 5:

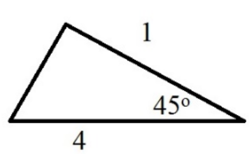
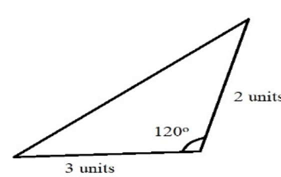


Figure 6:



25) Suppose that a right triangle has a hypotenuse of 5 and the other two sides sum to 7. What is the average length of the sides of the triangle?
   a) 7
   b) 4
   c) 3
   d) 5
   e) None of the above.

26) Suppose that a right triangle has a hypotenuse of $4\sqrt{5}$ and the other two sides sum to 12. What is the length of the shortest side of the triangle?
   a) $4\sqrt{5}$
   b) 8
   c) 4
   d) $\sqrt{5}$
   e) None of the above.

27) Consider the trapezoid in the *xy*-plane with vertices (0,0), (1,1), (2,1), and (2,0). Identify the point where the two diagonals of the trapezoid intersect.
   a) $\left(\frac{4}{3},\frac{2}{3}\right)$
   b) $\left(\frac{4}{3},\frac{4}{3}\right)$
   c) $\left(\frac{1}{3},\frac{4}{3}\right)$
   d) $\left(\frac{2}{3},\frac{2}{3}\right)$
   e) None of the above.

28) Select the statement which is FALSE.
   a) It is possible that two linear equations (of the form $Ax + By = C$, where *A, B,* and *C* are constants) will yield intersecting lines with equivalent slopes.
   b) Given a square and a circle with the same perimeter (circumference), the circle has a greater area.
   c) When two planes in 3D space intersect, their intersection is a single point.
   d) Triangles always have 3 vertices.
   e) None of the above.

29) A convex polygon is defined to be a polygon where every point on every line segment connecting every pair of points inside or on the boundary of the polygon remains inside or on the boundary of the polygon. Identify which of the following statements is FALSE.
   a) All triangles are convex polygons.
   b) All quadrilaterals are convex polygons.
   c) Each interior angle of a convex polygon is less than 180 degrees.
   d) For each edge of a convex polygon, all interior points are on the same side of the line that defines that edge.
   e) None of the above.

30) Refer to Figure 7. Suppose that A, B, and C are three points on the circle with radius r and center O. Further assume that $\triangle\ OAB$ and $\triangle\ OBC$ are equilateral triangles. What is the combined area of the two shaded segments? Remember, do not assume the figure is drawn to scale.

a) $\frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}$
b) $\frac{\pi r^2}{3} - \frac{\sqrt{3}r^2}{2}$
c) $\frac{\pi r^2}{6} + \frac{\sqrt{3}r^2}{4}$
d) $\frac{\pi r^2}{6}$
e) None of the above.

31) Assume that the triangle shown in Figure 8 is drawn to scale. Find the area of the region enclosed by the quadrilateral.

a) 2
b) 2.5
c) 3.5
d) 4
e) None of the above.

32) You do not know the meaning of "3 =1" or "4 = 2" other than they are logical statements. Which of the following is logically equivalent to the statement "If 3 = 1, then 4 = 2" ?

a) "If not 4 = 2, then not 3 =1"
b) "If 4 = 2 then 3 = 1"
c) "4 = 2 and 3 = 1"
d) "3 = 1 implies three is an odd number"
e) None of the above.

Figure 7:

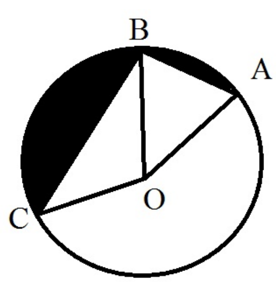


Figure 8: