# Indiana State Math Contest

# 2023

# Comprehensive

# Exam

This test was prepared by faculty of
Indiana University Purdue University Columbus
Rodney Lynch, Ph.D.

Mark your calendar:

*ICTM State Awards Ceremony 2023: Friday, June 9, 2023*
*ICTM State Math Contest 2024: Saturday, April TBA, 2024*

Do not open this test booklet until you have been advised to do so by the test proctor.

1. The area of a rectangle is numerically equal to its perimeter. If the length of the rectangle is 3 more than its width, find the area of the rectangle.

   (a) 40

   (b) 10

   (c) 18

   (d) 54

   (e) 28

2. If $i = \sqrt{-1}$ simplify

$$\frac{7+4i}{3-2i} + \frac{7-4i}{3+2i}.$$

   (a) $\frac{16}{65}$

   (b) 4

   (c) $\frac{58}{5}$

   (d) 2

   (e) $\frac{58}{13}$

3. One leg of a right triangle has length $\frac{24}{5}$. The area of the triangle is 7. Find the length of the hypotenuse.

   (a) $\frac{337}{60}$

   (b) $\frac{132}{25}$

   (c) $\frac{21}{4}$

   (d) $\frac{1189}{225}$

   (e) $\frac{238}{45}$

4. Write as a single root: $\sqrt[3]{4} \cdot \sqrt{3}$

   (a) $\sqrt[6]{108}$

   (b) $\sqrt[6]{2304}$

   (c) $\sqrt[6]{432}$

   (d) $\sqrt[6]{72}$

   (e) $\sqrt[6]{144}$

5. A water tank can be filled by two different pipes. If the drain is closed, it takes the first pipe $1\frac{1}{4}$ hours by itself to fill the tank and it takes the second pipe $1\frac{1}{5}$ hours by itself to fill the tank. If the tank is full, it takes $3\frac{1}{3}$ hours for the tank to empty with the drain opened. How long will it take to fill an empty tank if both pipes are used and the drain is opened?

   (a) 1 hour

   (b) $1\dfrac{1}{3}$ hours

   (c) $\dfrac{5}{6}$ hours

   (d) 2 hours

   (e) $\dfrac{3}{4}$ hours

6. Let $f(x) = 3x^3 - 9x$. Simplify $\dfrac{f(x) - f(1)}{(x-1)^2}$.

   (a) $3x + 6$

   (b) $3x + 2$

   (c) $3x - 8$

   (d) $3x + 4$

   (e) $3x - 12$

7. Find the positive integer $a$ which satisfies

$$\sqrt{a} + \sqrt{1183} = \sqrt{2023}.$$

(a) 98
(b) 840
(c) 252
(d) 112
(e) 63

8. If $n$ is a positive integer then

$$n! = n(n-1)(n-2) \cdot \ldots \cdot 1.$$

Simplify the expression $\dfrac{2021! + 2022!}{2023!}$.

(a) $\dfrac{1}{2023}$
(b) $\dfrac{2021}{2022}$
(c) $\dfrac{1}{2022}$
(d) $\dfrac{2023}{2022}$
(e) $\dfrac{2021}{2023}$

9. Three of the vertices of a parallelogram are at the points $(2, 5)$, $(3, 2)$, and $(7, 9)$. Which of the following sets consists of all possibilities for the fourth vertex?

(a) $\{(-3, 1), (6, 12), (8, 6)\}$
(b) $\{(-3, 1), (1, 8), (12, 13)\}$
(c) $\{(-1, -5), (4, -1), (11, 16)\}$
(d) $\{(-2, -2), (6, 12), (8, 6)\}$
(e) $\{(-1, -5), (6, 12), (12, 13)\}$

10. Find all values of $x$ which satisfy

$$|\, x^2 + 5x + 1 \,| = 2x + 5.$$

(a) $\{-6, -4\}$

(b) $\{-6, 1\}$

(c) $\{-1, 1\}$

(d) $\{-1, 2\}$

(e) $\{1, -4\}$

11. Which of the following is equal to $2\sin^{-1}\dfrac{15}{17}$?

(a) $\sin^{-1}\left(\dfrac{240}{289}\right)$

(b) $\sin^{-1}\left(\dfrac{161}{289}\right)$

(c) $\cos^{-1}\left(-\dfrac{161}{289}\right)$

(d) $\sin^{-1}\left(-\dfrac{240}{289}\right)$

(e) $\cos^{-1}\left(-\dfrac{240}{289}\right)$

12. Six people which I will call $A$, $B$, $C$, $D$, $E$, and $F$, go to the movies. How many different ways can all six people sit in the first six seats of a row if $A$ must sit next to $B$, $C$ must sit next to $D$, and $E$ must sit next to $F$?

(a) 60

(b) 48

(c) 36

(d) 72

(e) 120

13. Find the coordinates of the foci of the ellipse

$$\frac{(x+2)^2}{24} + \frac{(y-1)^2}{25} = 1.$$

(a) $(-2, 2)$, $(-2, 0)$

(b) $(-3, 1)$, $(-1, 1)$

(c) $(-2, 8)$, $(-2, -6)$

(d) $(-9, 1)$, $(5, 1)$

(e) $(-2 - 2\sqrt{6}\,, 1)$, $(-2 + 2\sqrt{6}\,, 1)$

14. Multiply and simplify:

$$(\,8 - 3\sqrt{5} + 4\sqrt{3} - 2\sqrt{15}\,)(\,8 - 3\sqrt{5} - 4\sqrt{3} + 2\sqrt{15}\,)$$

(a) $-89 + 12\sqrt{15}$

(b) $1$

(c) $7 - 6\sqrt{15}$

(d) $-12 - 6\sqrt{5} + 8\sqrt{3} - 4\sqrt{15}$

(e) $31$

15. Find the set of all real numbers $x$ which satisfy

$$\left(\frac{x}{x+2}\right)^2 \geq 9.$$

Write the answer using interval notation.

(a) $(-\infty, -3] \cup \left[-\frac{3}{2}, \infty\right)$

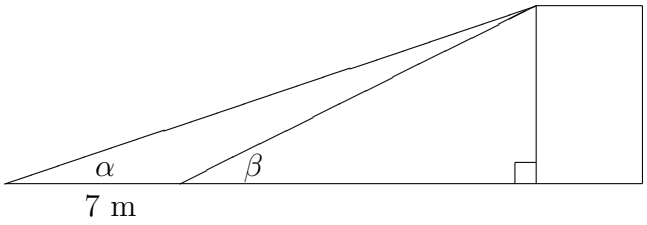
(b) $[-3, -2) \cup \left(-2, -\frac{3}{2}\right]$

(c) $[-3, -2) \cup \left[-\frac{3}{2}, \infty\right)$

(d) $(-\infty, -3] \cup \left(-2, -\frac{3}{2}\right]$

(e) $(-\infty, -2) \cup (-2, \infty)$

16. From a point on the ground, the angle of elevation to the top of a building is $\alpha = \tan^{-1}\frac{3}{4}$. From a point on the ground 7 meters closer to the building, the angle of elevation to the top of the building is $\beta = \tan^{-1}\frac{4}{3}$. Find the height of the building.




(a) 17 meters

(b) 25 meters

(c) 29 meters

(d) 21 meters

(e) 12 meters

17. How many sets $A$ satisfy

$$A \cap \{\,1,3,4,5\,\} = \{\,1,4\,\} \quad \text{and} \quad A \cup \{\,2,3,5\,\} = \{\,1,2,3,4,5,6\,\}\,?$$

(a) 2

(b) 3

(c) 1

(d) 4

(e) 0

18. Suppose $a$ and $b$ are real numbers with $a > 1$ that satisfy $\log_4(\log_2 a) = b$. Find the value of $\log_2(\log_4 a)$.

(a) $b - 1$

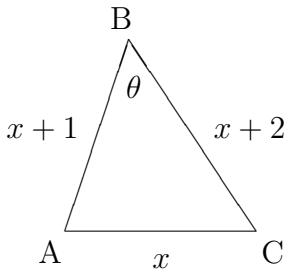
(b) $\frac{1}{2}\,b$

(c) $2b - 1$

(d) $4b - 1$

(e) $\frac{1}{2}\,b + \frac{1}{2}$

19. In the triangle below, $\theta = \cos^{-1}\dfrac{2}{3}$. Find $x$.




(a) 6

(b) 8

(c) 4

(d) 5

(e) 7

20. Let $f(x) = \begin{cases} -2x + 3 & \text{if} \quad x \leq 3, \\ 4x - 15 & \text{if} \quad x > 3. \end{cases}$

Find the number of points of intersection of the graphs of

$$y = f(x) \quad \text{and} \quad y = x^2 - 4x.$$

(a) 4

(b) 3

(c) 1

(d) 0

(e) 2

21. Let $A$ and $B$ be the following $2 \times 2$ matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

Find a number $c$ with the property $BA = cAB$.

(a) $2$

(b) $1$

(c) $-1$

(d) $0$

(e) $\dfrac{1}{2}$

22. Find the smallest positive number $k$ for which the graphs of

$$y = \cos\left(\frac{\pi}{4}\,x + k\right) \quad \text{and} \quad y = \sin\left(\frac{\pi}{4}\,x\right)$$

are identical.

(a) $\dfrac{\pi}{2}$

(b) $\dfrac{3\pi}{2}$

(c) $\dfrac{3\pi}{4}$

(d) $\pi$

(e) $\dfrac{\pi}{4}$

23. Find the point of intersection of the lines $l_1$ and $l_2$ whose parametric equations are given by

$$l_1 : \; x = 2t + 2, \;\; y = 2t, \;\; z = -2t - 3,$$

$$l_2 : \; x = 2t - 1, \;\; y = t - 1, \;\; z = -t - 2.$$

(a) $(2, 0, -3)$

(b) $(3, 1, -4)$

(c) $(5, 3, -6)$

(d) $(1, -1, -2)$

(e) $(4, 2, -5)$

24. Four six-sided dice are rolled. Find the probability that all dice came up 3 or greater.

(a) $\dfrac{5}{432}$

(b) $\dfrac{11}{648}$

(c) $\dfrac{19}{72}$

(d) $\dfrac{16}{81}$

(e) $\dfrac{1}{54}$

25. Find the $2 \times 2$ matrix $A$ which satisfies

$$A^2 = \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} \quad \text{and} \quad A^3 = \begin{bmatrix} 1 & 18 \\ -18 & 19 \end{bmatrix}.$$

(a) $\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

26. Let $f(x) = x^2 + 2x - 2$. Find all values of $x$ which satisfy

$$f(f(x)) = (f(x))^2.$$

(a) $\left\{6 - 2\sqrt{3},\, 6 + 2\sqrt{3}\,\right\}$

(b) $\{-6, 1\}$

(c) $\left\{4 - 2\sqrt{3},\, 4 + 2\sqrt{3}\,\right\}$

(d) $\{-3,\, 1\}$

(e) $\{-8, 12\}$

27. There exist unique real numbers $a$, $b$, $c$, and $d$ such that the polynomial $1+x+x^2+x^3$ can be written in the form

$$1+x+x^2+x^3 = a+bx+cx(x-1)+dx(x-1)(x-2).$$

Which of the following is equal to $a+b+c+d$?

(a) 7
(b) 6
(c) 8
(d) 5
(e) 9

28. There exist unique positive integers $a$, $b$, $c$, and $d$ with

$$\frac{1000}{2023} = \cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}}.$$

Find $d$.

(a) 15
(b) 19
(c) 23
(d) 17
(e) 11

29. A point $P$ is chosen on the part of the graph of $y = 4-x^2$ that lies in the first quadrant. Consider the quadrilateral that has vertices

$$(0,0),\ (0,4),\ (2,0),\ \text{ and }\ P.$$

Find the coordinates of the point $P$ that makes the quadrilateral of largest area.

(a) $(1,3)$
(b) $\left(\frac{2}{3}, \frac{32}{9}\right)$
(c) $\left(\frac{1}{2}, \frac{15}{4}\right)$
(d) $\left(\frac{4}{3}, \frac{20}{9}\right)$
(e) $\left(\frac{3}{2}, \frac{7}{4}\right)$

30. Let $f(x) = \begin{cases} -2x+8 & \text{if} \quad x \le 3, \\ -\frac{1}{2}\,x + \frac{7}{2} & \text{if} \quad x > 3. \end{cases}$

Find a formula for $f^{-1}(x)$.

(a) $f^{-1}(x) = \begin{cases} -\frac{1}{2}\,x + 4 & \text{if} \quad x < 2, \\ -2x+7 & \text{if} \quad x \ge 2. \end{cases}$

(b) $f^{-1}(x) = \begin{cases} -2x+7 & \text{if} \quad x < 3, \\ -\frac{1}{2}\,x + 4 & \text{if} \quad x \ge 3. \end{cases}$

(c) $f^{-1}(x) = \begin{cases} -\frac{1}{2}\,x + 4 & \text{if} \quad x < 3, \\ -2x+7 & \text{if} \quad x \ge 3. \end{cases}$

(d) $f^{-1}(x) = \begin{cases} -2x+7 & \text{if} \quad x < 2, \\ -\frac{1}{2}\,x + 4 & \text{if} \quad x \ge 2. \end{cases}$

(e) $f^{-1}(x) = \begin{cases} -\frac{1}{2}\,x + \frac{7}{2} & \text{if} \quad x < 3, \\ -2x+8 & \text{if} \quad x \ge 3. \end{cases}$

31. Find the exact value of the sum

$$\sum_{n=0}^{\infty} (-1)^{n(n-1)/2} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \frac{1}{32} - \cdots .$$

(a) $\dfrac{6}{5}$

(b) $\dfrac{2}{3}$

(c) $2$

(d) $\dfrac{2}{15}$

(e) $\dfrac{2}{5}$

32. Let $a$, $b$, $c$ be real numbers with $a \neq 0$ and let $r_1$ and $r_2$ be roots of the quadratic equation $ax^2 + bx + c = 0$. Which of the following quadratic equations has roots $r_1^2$ and $r_2^2$?

(a) $a^2x^2 + (4ac - b^2)x + c^2 = 0$

(b) $a^2x^2 + b^2x + c^2 = 0$

(c) $a^2x^2 + (b^2 - 4ac)x + c^2 = 0$

(d) $a^2x^2 + (b^2 - 2ac)x + c^2 = 0$

(e) $a^2x^2 + (2ac - b^2)x + c^2 = 0$

33. The line $y = x+4$ is tangent to the parabola $y = x^2+ax+b$ at the point $(2, 6)$. Find $b$.

(a) 16

(b) $-4$

(c) 12

(d) $-2$

(e) 8

34. Find the distance between the parallel planes

$$2x + 3y + 6z = 1925 \quad \text{and} \quad 2x + 3y + 6z = 2023.$$

(a) 49

(b) 56

(c) 14

(d) 98

(e) 63

35. Which of the following is a root of

$$4x^3 - 6x^2 + 4x - 1 = 0\,?$$

(a) $-\dfrac{1}{2} + \dfrac{\sqrt{2}}{2}\,i$

(b) $-\dfrac{1}{2} + \dfrac{1}{2}\,i$

(c) $\dfrac{1}{2} + \dfrac{1}{2}\,i$

(d) $-\dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\,i$

(e) $\dfrac{1}{2} + \dfrac{\sqrt{2}}{2}\,i$

36. Find all values of $\theta$ in the interval $[0, 2\pi)$ which satisfy

$$\sin\left(2\theta + \frac{\pi}{3}\right) = \sin\left(\theta - \frac{\pi}{4}\right).$$

(a) $\left\{ \dfrac{7\pi}{36}, \dfrac{31\pi}{36}, \dfrac{17\pi}{12}, \dfrac{55\pi}{36} \right\}$

(b) $\left\{ \dfrac{\pi}{4}, \dfrac{33\pi}{36}, \dfrac{13\pi}{12}, \dfrac{19\pi}{12} \right\}$

(c) $\left\{ \dfrac{11\pi}{36}, \dfrac{35\pi}{36}, \dfrac{17\pi}{12}, \dfrac{59\pi}{36} \right\}$

(d) $\left\{ \dfrac{\pi}{12}, \dfrac{3\pi}{4}, \dfrac{43\pi}{36}, \dfrac{17\pi}{12} \right\}$

(e) $\left\{ \dfrac{5\pi}{36}, \dfrac{29\pi}{36}, \dfrac{13\pi}{12}, \dfrac{53\pi}{36} \right\}$