Indiana State Math Contest

2019

Comprehensive Exam

This test was prepared by faculty of

Indiana University Purdue University Columbus

Rodney Lynch, Ph.D.

Do not open this test booklet until you have been advised to do so by the test proctor.

1. Let $f(x) = x^2 + \dfrac{1}{x^2}$. Compute the exact value of $f(8 + 3\sqrt{7})$.

   (a) $217 + 14\sqrt{7}$

   (b) $201 + 20\sqrt{7}$

   (c) 258

   (d) 16130/127

   (e) 254

2. Which of the equations below describe the set of all points (x, y) in the plane that are equidistant from the points $(-1, 8)$ and $(4, -7)$?

   (a) $x - 3y = 0$

   (b) $3x + y = 5$

   (c) $x - 6y = 1$

   (d) $x + 3y = 3$

   (e) $3x - y = -3$

3. A chemist has a solution that is 60% acid and a solution that is 30% acid. How many milliliters of each are needed to be mixed together to create 42 milliliters of a solution that is 52% acid?

   (a) 31.6 ml of 60% acid, 10.4 ml of 30% acid

   (b) 33.2 ml of 60% acid, 8.8 ml of 30% acid

   (c) 32.4 ml of 60% acid, 9.6 ml of 30% acid

   (d) 30.8 ml of 60% acid, 11.2 ml of 30% acid

   (e) 34 ml of 60% acid, 8 ml of 30% acid

4. Find the set of all complex numbers x which satisfy

$$x^2 + (4 + 2\sqrt{3})x + 4\sqrt{3} = 0.$$

   (a) $\{-4 - 2\sqrt{3} + \sqrt{7},\ -4 - 2\sqrt{3} + \sqrt{7}\}$

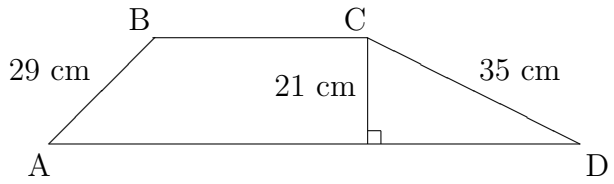
   (b) $\{-2 - \sqrt{3} + \sqrt{7},\ -2 - \sqrt{3} - \sqrt{7}\}$

   (c) $\{-2 - \sqrt{3} + \sqrt{7 - 4\sqrt{3}},\ -2 - \sqrt{3} - \sqrt{7 - 4\sqrt{3}}\}$

   (d) $\{-4 - 2\sqrt{3} + \sqrt{7 - 4\sqrt{3}},\ -4 - 2\sqrt{3} - \sqrt{7 - 4\sqrt{3}}\}$

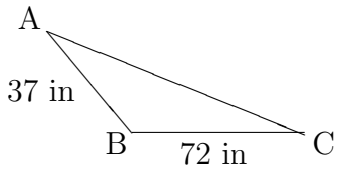
   (e) $\{-4 - \sqrt{3} + \sqrt{7},\ -4 - \sqrt{3} + \sqrt{7}\}$

5. The perimeter of the trapezoid $ABCD$ pictured below is 160 cm. Find the area of trapezoid $ABCD$.



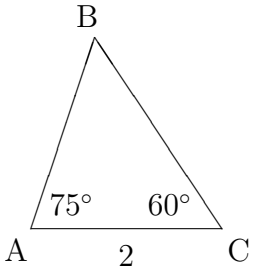
(a) 840 cm^2
(b) 1680 cm^2
(c) 1008 cm^2
(d) 756 cm^2
(e) 945 cm^2

6. In the picture below, $\triangle ABC$ has area 1260 in^2. Find the length of side AC.



(a) 86 in
(b) 84 in
(c) 101 in
(d) 91 in
(e) 97 in

7. Find the length of side AB in the triangle below.



(a) $\sqrt{2}+\sqrt{6}$
(b) $\sqrt{6}-\sqrt{2}$
(c) $\sqrt{6}$
(d) $8\sqrt{3}\,/\,3$
(e) $\sqrt{3}+\sqrt{6}$

8. Let $f(x) = \dfrac{x+1}{2x+1}$. Simplify $f(f(x))$.

   (a) $\dfrac{3x+2}{4x+3}$

   (b) $\dfrac{2x+3}{3x+4}$

   (c) $\dfrac{x+2}{2x+3}$

   (d) x

   (e) $\dfrac{2x+1}{3x+2}$

9. Which of the following is the equation of the parabola that represents the set of all points (x, y) in the plane that are equidistant from the point $(3, -4)$ and the line $x = -2$?

   (a) $(x-3)^2 = -4(y+3)$

   (b) $(y+4)^2 = -10(x-\frac{1}{2})$

   (c) $(x-3)^2 = 4(y+3)$

   (d) $(y+4)^2 = 10(x-\frac{1}{2})$

   (e) $(x-3)^2 = -8(y+4)$

10. Find all solutions of

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$$

that are in the interval $[0, 2\pi]$.

   (a) $\left\{\dfrac{\pi}{12}, \dfrac{7\pi}{12}\right\}$

   (b) $\left\{\dfrac{\pi}{12}, \dfrac{13\pi}{12}\right\}$

   (c) $\left\{\dfrac{\pi}{12}, \dfrac{19\pi}{12}\right\}$

   (d) $\left\{\dfrac{\pi}{12}, \dfrac{11\pi}{12}\right\}$

   (e) $\left\{\dfrac{\pi}{12}, \dfrac{23\pi}{12}\right\}$

11. We are given three functions f, g, and h that each have domain and range equal to $\{1, 2, 3, 4\}$ and $h = f \circ g$. The functions f and h are given by

x	1	2	3	4
$f(x)$	3	1	4	2

,

x	1	2	3	4
$h(x)$	4	2	1	3

.

Which of the following is g?

(a)

x	1	2	3	4
$g(x)$	2	3	4	1

(b)

x	1	2	3	4
$g(x)$	4	1	2	3

(c)

x	1	2	3	4
$g(x)$	3	4	2	1

(d)

x	1	2	3	4
$g(x)$	4	3	1	2

(e)

x	1	2	3	4
$g(x)$	2	1	3	4

12. An urn contains 4 red and 5 yellow marbles. 2 marbles are selected without replacement. Find the probability that both marbles selected are the same color.

(a) 1/6
(b) 4/9
(c) 7/12
(d) 5/18
(e) 11/36

13. Find the range of the following function:

$$f(x) = -2 - \sqrt{8x - x^2}.$$

Express your answer using interval notation.

(a) $[-\frac{9}{4}, -2]$
(b) $[-4, -2]$
(c) $[-6, -2]$
(d) $[-\frac{9}{2}, -2]$
(e) $[-8, -2]$

14. A list of 2019 numbers has a mean of 2017. One of the numbers in the list is removed. The mean of the remaining numbers is 2015. What number was removed from the original list?

(a) 6053

(b) 4038

(c) 2016

(d) 4036

(e) 6048

15. Which of the following is the equation of an ellipse that passes through the point $(7, 3)$ and has foci $(-5, -6)$ and $(-5, -2)$?

(a) $\dfrac{(x+5)^2}{196} + \dfrac{(y+4)^2}{192} = 1$

(b) $\dfrac{(x+5)^2}{192} + \dfrac{(y+4)^2}{196} = 1$

(c) $\dfrac{(x+5)^2}{49} + \dfrac{(y+4)^2}{16} = 1$

(d) $\dfrac{(x-5)^2}{49} + \dfrac{(y-5)^2}{49} = 1$

(e) $\dfrac{(x+5)^2}{16} + \dfrac{(y+4)^2}{49} = 1$

16. The equation

$$(x+a)(x+b) = (c+a)(c+b)$$

has $x = c$ as one solution. Which of the following is the other solution?

(a) $x = a - b - c$

(b) $x = -a - b - c$

(c) $x = -a - b + c$

(d) $x = -a + b - c$

(e) $x = a + b - c$

17. Let A and B be the following 2×2 matrices:

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}.$$

Find a 2×2 matrix X with $XA = B$.

(a) $X = \begin{bmatrix} 16 & -1 \\ 10 & 0 \end{bmatrix}$

(b) $X = \begin{bmatrix} 9 & -16 \\ 13 & -22 \end{bmatrix}$

(c) $X = \begin{bmatrix} -9 & 16 \\ -13 & 22 \end{bmatrix}$

(d) $X = \begin{bmatrix} -24 & -11 \\ -14 & -6 \end{bmatrix}$

(e) $X = \begin{bmatrix} -16 & 1 \\ -10 & 0 \end{bmatrix}$

18. It can be shown that the following formula is true for all integers $n \geq 1$:

$$\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}.$$

Which of the following equals $\displaystyle\sum_{k=n+1}^{2n} k(k+1)$?

(a) $\dfrac{n(n+1)(7n+2)}{3}$

(b) $\dfrac{n(n+1)(3n+4)}{3}$

(c) $\dfrac{n(n+1)(3n+5)}{3}$

(d) $\dfrac{n(n+1)(5n+6)}{3}$

(e) $\dfrac{n(n+1)(9n+4)}{3}$

19. Compute the exact value of $\cos 18.75^\circ \cos 3.75^\circ + \sin 18.75^\circ \sin 3.75^\circ$.

(a) $\dfrac{\sqrt{2}}{2}$

(b) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

(c) $\dfrac{\sqrt{2+\sqrt{2}}}{2}$

(d) $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

(e) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

20. Consider the following equation:

$$\log_6(x^2+8x) + \log_6(x+3) = 2.$$

Which of the statements below is true of its solution set?

(a) It contains three distinct elements: three distinct positive integers.

(b) It contains only two distinct elements: two distinct positive integers.

(c) It contains only two distinct elements: a positive integer and a negative integer.

(d) It contains only one element: a positive integer.

(e) It is the empty set.

21. Find the area enclosed by the circle with polar equation $r = -2\sin\theta$.

(a) π

(b) $\frac{\pi}{4}$

(c) 4π

(d) $\pi\sqrt{2}$

(e) 2π

22. In a certain math class, the instuctor gives a quiz on Monday with probability 0.2, a quiz on Wednesday with probability 0.3, and a quiz on Friday with probability 0.6. Assume these events are independent. Find the probability of having exactly one quiz in a given week.

(a) 0.488

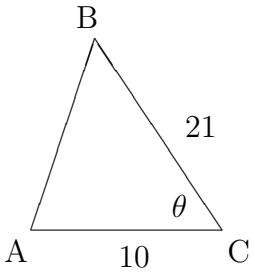
(b) 0.776

(c) 0.748

(d) 0.964

(e) 0.252

23. In the triangle below, $\theta = \tan^{-1}\frac{4}{3}$. Find the length of side AB.



(a) 19

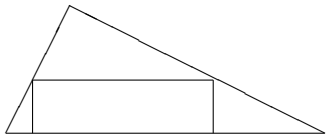
(b) $\sqrt{205}$

(c) 17

(d) 15

(e) 23

24. A rectangle is inscribed in a triangle with a base of 10 inches and a height of 8 inches. Find the largest area of such a rectangle.



(a) 28 in^2

(b) 32 in^2

(c) 25 in^2

(d) 20 in^2

(e) 24 in^2

25. Find the set of all real numbers x which satisfy the inequality

$$|x^2 - 13x| < 30.$$

Express your answer using interval notation.

(a) $(-\infty, 3) \cup (10, 15)$

(b) $(-2, 3) \cup (10, 15)$

(c) $(-2, 10) \cup (15, \infty)$

(d) $(-\infty, 0) \cup (13, 15)$

(e) $(-2, 0) \cup (13, 15)$

26. Write the following function in piecewise form: $f(x) = |x + 2| - 2|x + 1| + |x|$.

(a) $f(x) = \begin{cases} 0 & \text{if} \quad x \leq -2, \\ 2x + 4 & \text{if} \quad -2 < x \leq -1, \\ -2x & \text{if} \quad -1 < x \leq 0, \\ 0 & \text{if} \quad x > 0. \end{cases}$

(b) $f(x) = \begin{cases} 2x & \text{if} \quad x \leq -2, \\ -4 & \text{if} \quad -2 < x \leq -1, \\ 2x - 2 & \text{if} \quad -1 < x \leq 0, \\ -2x & \text{if} \quad x > 0. \end{cases}$

(c) $f(x) = \begin{cases} -2x & \text{if} \quad x \leq -2, \\ 4 & \text{if} \quad -2 < x \leq -1, \\ -2x + 2 & \text{if} \quad -1 < x \leq 0, \\ 2 & \text{if} \quad x > 0. \end{cases}$

(d) $f(x) = \begin{cases} -2x - 4 & \text{if} \quad x \leq -2, \\ 0 & \text{if} \quad -2 < x \leq 0, \\ 2x & \text{if} \quad x > 0. \end{cases}$

(e) $f(x) = \begin{cases} 0 & \text{if} \quad x \leq -2, \\ -2x - 4 & \text{if} \quad -2 < x \leq -1, \\ 2x & \text{if} \quad -1 < x \leq 0, \\ 0 & \text{if} \quad x > 0. \end{cases}$

27. Find the number of points of intersection of the parabolas with equations

$$x = y^2,$$

$$(x-3)^2 = -8\left(y - \frac{3}{2}\right).$$

(a) 1
(b) 2
(c) 4
(d) 3
(e) 0

28. In the plane, a point Q is said to be the *reflection of the point P about the line l* if the following conditions are true:

- The line segment PQ is perpendicular to the line l
- The midpoint of the line segment PQ lies on the line l

Find the coordinates of the point obtained by reflecting the point (a, b) about the line $y = -2x$.

(a) $\left(-\frac{1}{3}a - \frac{2}{3}b, -\frac{2}{3}a + \frac{1}{3}b\right)$
(b) $\left(-\frac{3}{5}a - \frac{4}{5}b, -\frac{4}{5}a + \frac{3}{5}b\right)$
(c) $\left(-\frac{1}{2}a - \frac{3}{4}b, -\frac{3}{4}a + \frac{1}{2}b\right)$
(d) $\left(-\frac{1}{2}a - \frac{3}{4}b, -\frac{1}{2}a + \frac{3}{4}b\right)$
(e) $\left(-\frac{3}{5}a - \frac{4}{5}b, -\frac{3}{5}a + \frac{4}{5}b\right)$

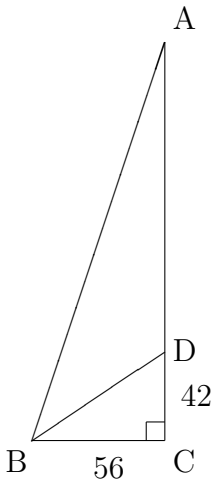
29. There exist unique real numbers a, b, c, and d such that the polynomial x^3 can be written in the form

$$x^3 = a + b(x+2) + c(x+2)^2 + d(x+2)^3.$$

Which of the following is equal to c?

(a) -12
(b) 12
(c) 6
(d) -6
(e) -4

30. In the picture below, $\angle ABD$ is congruent to $\angle DBC$ and the point D lies on the side AC. Find the length of AD.



(a) 63
(b) 348
(c) 48
(d) 740
(e) 150

31. Let $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$. Simplify $1 + w + w^2 + w^3 + \cdots + w^{2019}$.

(a) $-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i$
(b) 1
(c) $\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$
(d) 0
(e) $-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i$

32. How many positive integers less than 2019 are divisible by 5 or 7 but not by 17?

(a) 716
(b) 428
(c) 770
(d) 598
(e) 626