2019 Comprehensive Exam
Answer Key

1.	e	9.	d	17.	b	25.	b
2.	a	10.	c	18.	a	26.	a
3.	d	11.	c	19.	e	27.	b
4.	b	12.	b	20.	d	28.	b
5.	c	13.	c	21.	a	29.	d
6.	d	14.	a	22.	a	30.	e
7.	c	15.	b	23.	c	31.	b
8.	a	16.	b	24.	d	32.	d

2019 Comprehensive Exam Solutions

1. Note that $(8+3\sqrt{7})^2 = 64 + 48\sqrt{7} + 9\cdot 7 = 127 + 48\sqrt{7}$ and
$$\frac{1}{8+3\sqrt{7}} = \frac{8-3\sqrt{7}}{(8+3\sqrt{7})(8-3\sqrt{7})} = \frac{8-3\sqrt{7}}{64-9\cdot 7} = 8-3\sqrt{7}.$$
So we also have
$$\frac{1}{(8+3\sqrt{7})^2} = (8-3\sqrt{7})^2 = 127-48\sqrt{7}.$$
Hence
$$\begin{aligned} f(8+3\sqrt{7}) &= (8+3\sqrt{7})^2 + \frac{1}{(8+3\sqrt{7})^2} \\ &= 127+48\sqrt{7}+127-48\sqrt{7} \\ &= 254. \end{aligned}$$

2. **Solution 1**: Let (x,y) be a point in the plane that is equidistant from each of the given points. Using the distance formula,
$$\sqrt{(x+1)^2+(y-8)^2} = \sqrt{(x-4)^2+(y+7)^2}.$$
Squaring both sides we obtain
$$x^2+2x+1+y^2-16y+64 = x^2-8x+16+y^2+14y+49$$
Cancelling the x^2 and y^2 terms on both sides and simplifying gives
$$\begin{aligned} 2x-16y+65 &= -8x+14y+65 \\ 10x-30y &= 0 \\ x-3y &= 0. \end{aligned}$$
Solution 2: The set of all points in the plane equidistant from two given points is the perpendicular bisector of the line segment joining the two points. The midpoint of the line segment joining the points $(-1,8)$, $(4,-7)$ is $(\frac{-1+4}{2}, \frac{8-7}{2}) = (\frac{3}{2}, \frac{1}{2})$. The slope of the line segment joining the two points is $\frac{-7-8}{4+1} = \frac{-15}{5} = -3$. So the perpendicular bisector has slope $m = \frac{1}{3}$ and passes through the point $(\frac{3}{2}, \frac{1}{2})$. Its equation in point slope form is given by
$$\begin{aligned} y - \frac{1}{2} &= \frac{1}{3}\left(x - \frac{3}{2}\right) \\ y - \frac{1}{2} &= \frac{1}{3}x - \frac{1}{2} \\ y &= \frac{1}{3}x \quad \Rightarrow \quad x-3y=0. \end{aligned}$$

3. Let $x =$ the number of ml of 60% acid solution needed and $y =$ the number of ml of 30% acid solution needed. We obtain the following system of equations:

$$\begin{aligned} x + y &= 42 \\ .6x + .3y &= .52 \cdot 42 = 21.84 \end{aligned}$$

Multiplying the first equation by $-.3$ and adding it to the second gives

$$.3x = 9.24 \quad \Rightarrow \quad x = 30.8 \text{ ml } 60\% \text{ acid.}$$

We also get $y = 42 - 30.8 = 11.2$ ml 30% acid.

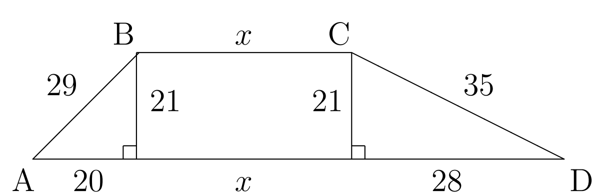
4. We can use the quadratic formula with $a = 1$, $b = 4 + 2\sqrt{3}$, and $c = 4\sqrt{3}$. Note that

$$\begin{aligned} b^2 - 4ac &= (4 + 2\sqrt{3})^2 - 4 \cdot 1 \cdot 4\sqrt{3} \\ &= 16 + 16\sqrt{3} + 4 \cdot 3 - 16\sqrt{3} \\ &= 28. \end{aligned}$$

So the solutions are given by

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(4 + 2\sqrt{3}) \pm \sqrt{28}}{2} \\ &= \frac{-4 - 2\sqrt{3} \pm 2\sqrt{7}}{2} \\ &= -2 - \sqrt{3} \pm \sqrt{7}. \end{aligned}$$

5. First make use of the Pythagorean Theorem:



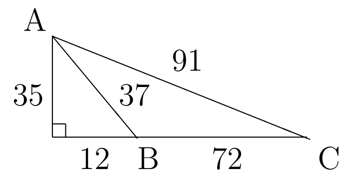
Since the perimeter is 160 cm we have:

$$2x + 112 = 160 \quad \Rightarrow \quad x = 24 \text{ cm.}$$

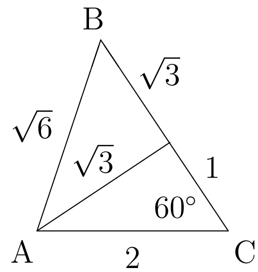
So the area of the trapezoid is given by

$$A = \frac{h}{2}(b_1 + b_2) = \frac{21}{2}((20 + 24 + 28) + 24) = \frac{21}{2} \cdot 96 = 1008 \text{ cm}^2.$$

6. Begin by dropping an altitude from vertex A to the opposite side BC. I'll denote the length of the altitude by h. Note that this is the height of the triangle, so since the area is 1260 in^2 we have $\frac{1}{2} \cdot 72h = 1260 \;\Rightarrow\; h = 35$ in. Now make use of the Pythagorean Theorem:



7. **Solution 1**: Begin by dropping an altitude from vertex A to the opposite side BC. Then make use of properties of 30-60-90 and 45-45-90 triangles:



Solution 2: Let x be the length of side AB. By the law of sines,

$$\frac{\sin 45^\circ}{2} = \frac{\sin 60^\circ}{x} \qquad \Rightarrow \qquad x = \frac{2\sin 60^\circ}{\sin 45^\circ} = \frac{2 \cdot \frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}}} = \sqrt{3} \cdot \sqrt{2} = \sqrt{6}.$$

8. $$f(f(x)) = \frac{\frac{x+1}{2x+1} + 1}{2 \cdot \frac{x+1}{2x+1} + 1} = \frac{x+1+2x+1}{2(x+1)+2x+1} = \frac{3x+2}{4x+3}.$$

9. The line $x = -2$ is the directrix and the point $(3, -4)$ is the focus of the parabola. Since the directrix is vertical and the focus is to the right of the directrix, the parabola opens to the right. The vertex of the parabola is located halfway between the directrix and the focus on the line $y = -4$, so the vertex is $(\frac{-2+3}{2}, -4) = (\frac{1}{2}, -4)$. The distance from the focus to the vertex is $3 - \frac{1}{2} = \frac{5}{2}$, so the equation of the parabola is given by

$$(y+4)^2 = 4 \cdot \frac{5}{2}\left(x - \frac{1}{2}\right) \;\Rightarrow\; (y+4)^2 = 10\left(x - \frac{1}{2}\right).$$

10. Let $\theta = x + \frac{\pi}{6}$. Since $0 \le x \le 2\pi$, we have $\frac{\pi}{6} \le x + \frac{\pi}{6} \le \frac{13\pi}{6}$. So we need to find the solutions of the equation $\cos\theta = \frac{\sqrt{2}}{2}$ that are in the interval $[\frac{\pi}{6}, \frac{13\pi}{6}]$. We get $\theta = \frac{\pi}{4}$ and $\theta = \frac{7\pi}{4}$. This gives

$$x = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}, \quad x = \frac{7\pi}{4} - \frac{\pi}{6} = \frac{19\pi}{12}.$$

11. Since the function f is one-to-one, it has an inverse f^{-1} found by swapping all of the entries in the ordered pairs of f:

x	1	2	3	4
$f^{-1}(x)$	2	4	1	3

Now $h = f \circ g \;\Rightarrow\; f^{-1} \circ h = f^{-1} \circ f \circ g \;\Rightarrow\; f^{-1} \circ h = g$. So we obtain:

$$g(1) = f^{-1}(h(1)) = f^{-1}(4) = 3, \quad g(2) = f^{-1}(h(2)) = f^{-1}(2) = 4,$$
$$g(3) = f^{-1}(h(3)) = f^{-1}(1) = 2, \quad g(4) = f^{-1}(h(4)) = f^{-1}(3) = 1.$$

12. **Solution 1**:

$$\begin{aligned} P(\text{both same color}) &= P(\text{both red}) + P(\text{both yellow}) \\ &= \frac{\binom{4}{2}}{\binom{9}{2}} + \frac{\binom{5}{2}}{\binom{9}{2}} \\ &= \frac{6}{36} + \frac{10}{36} = \frac{4}{9}. \end{aligned}$$

Solution 2:

$$\begin{aligned} P(\text{both same color}) &= P(\text{both red}) + P(\text{both yellow}) \\ &= P(\text{first red}) \cdot P(\text{second red} \,|\, \text{first red}) \\ &\qquad + P(\text{first yellow}) \cdot P(\text{second yellow} \,|\, \text{first yellow}) \\ &= \frac{4}{9} \cdot \frac{3}{8} + \frac{5}{9} \cdot \frac{4}{8} = \frac{4}{9}. \end{aligned}$$

13. First replace $f(x)$ by y: $y = -2 - \sqrt{8x - x^2} \;\Rightarrow\; y + 2 = -\sqrt{8x - x^2}$. Now square both sides, rearrange terms, and complete the square:

$$\begin{aligned} (y+2)^2 &= 8x - x^2 \\ x^2 - 8x + (y+2)^2 &= 0 \\ x^2 - 8x + 16 + (y+2)^2 &= 16 \\ (x-4)^2 + (y+2)^2 &= 16. \end{aligned}$$

So the graph of $f(x)$ is the lower semicircle of the above equation of a circle. The semicircle is centered at $(4, -2)$ and has radius 4. So the range of $f(x)$ is $[-6, -2]$.

14. Let x be the number removed from the original list. Note that when you remove one number you'll have 2018 remaining. The sum of all 2019 numbers is $2019 \cdot 2017$, the sum of the numbers remaining after removing x is $2018 \cdot 2015$, so

$$x = 2019 \cdot 2017 - 2018 \cdot 2015 = 6053.$$

15. The ellipse is centered at the midpoint of the line segment joining the foci, so the center of the ellipse is $(-5, \frac{-6-2}{2}) = (-5, -4)$. The distance from the center to a foci is $c = 2$. An ellipse is the set of all points in the plane for which the sum of the distances from two fixed points (the foci) is constant. The constant is in fact $2a$ where a is the distance from the center of the ellipse to a vertex on the major axis. Since $(7, 3)$ is a point on the ellipse, we compute the distances from this point to each foci:

$$\begin{aligned} 2a &= \sqrt{(7+5)^2 + (3+6)^2} \;+\; \sqrt{(7+5)^2 + (3+2)^2} \\ &= \sqrt{144+81} + \sqrt{144+25} \\ &= \sqrt{225} + \sqrt{169} \\ &= 28. \end{aligned}$$

Hence $a = 14$. For an ellipse, $c^2 = a^2 - b^2$ where b is the distance from the center to an endpoint of the minor axis.

$$2^2 = 14^2 - b^2 \;\Rightarrow\; b^2 = 14^2 - 2^2 = 192.$$

The foci lie on the major axis, so the major axis in this case is vertical. The form of the equation of an ellipse centered at the origin with vertical major axis is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. In this case, since the ellipse is centered at $(-5, -4)$, its equation is given by

$$\frac{(x+5)^2}{192} + \frac{(y+4)^2}{196} = 1.$$

16. First multiply it out, then get all terms on one side:

$$\begin{aligned} (x+a)(x+b) &= (c+a)(c+b) \\ x^2 + bx + ax + ab &= c^2 + bc + ac + ab \\ x^2 - c^2 + bx - bc + ax - ac &= 0 \\ (x+c)(x-c) + b(x-c) + a(x-c) &= 0 \\ (x-c)(x+c+b+a) &= 0 \end{aligned}$$

So the other solution is $x = -a - b - c$.

17. Begin by multiplying both sides on the right by A^{-1}: $XAA^{-1} = BA^{-1} \;\Rightarrow\; X = BA^{-1}$. Note that $\det(A) = 2 \cdot (-3) - (-5) \cdot 1 = -1$, so using the formula for the inverse of a 2×2 matrix we obtain

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}.$$

So

$$X = BA^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -16 \\ 13 & -22 \end{bmatrix}.$$

18. First observe $\sum_{k=n+1}^{2n} k(k+1) = \sum_{k=1}^{2n} k(k+1) - \sum_{k=1}^{n} k(k+1)$. The first sum on the right is evaluated by replacing n with $2n$ in the given formula. So we have

$$\begin{aligned}\sum_{k=n+1}^{2n} k(k+1) &= \frac{2n(2n+1)(2n+2)}{3} - \frac{n(n+1)(n+2)}{3} \\ &= \frac{4n(2n+1)(n+1)}{3} - \frac{n(n+1)(n+2)}{3} \\ &= \frac{n(n+1)}{3} \cdot (4(2n+1) - (n+2)) \\ &= \frac{n(n+1)(7n+2)}{3}.\end{aligned}$$

19. Make use of addition formulas:

$$\begin{aligned}\cos 18.75^\circ \cos 3.75^\circ + \sin 18.75^\circ \sin 3.75^\circ &= \cos(18.75^\circ - 3.75^\circ) \\ &= \cos 15^\circ \\ &= \cos(45^\circ - 30^\circ) \\ &= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

20. First solve the equation:

$$\begin{aligned}\log_6(x^2+8x) + \log_6(x+3) &= 2 \\ \log_6(x^2+8x)(x+3) &= 2 \\ 6^2 &= x^3 + 11x^2 + 24x \\ x^3 + 11x^2 + 24x - 36 &= 0.\end{aligned}$$

By inspection, $x = 1$ is a root of the above equation, so $x - 1$ is a factor. Using synthetic division we can obtain the other factor, so the equation becomes

$$\begin{aligned}(x-1)(x^2+12x+36) &= 0 \\ (x-1)(x+6)^2 &= 0.\end{aligned}$$

This has $x = 1$ and $x = -6$ as solutions. But note that $x = -6$ does not work in the original equation since we cannot take logarithms of negative numbers. So $x = 1$ is the only solution, hence the solution set contains only one element: a positive integer.

21. First find a Cartesian equation for the circle. Use $y = r\sin\theta$:

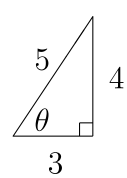
$$\begin{aligned} r &= -2\sin\theta \\ r &= -2\cdot\frac{y}{r} \\ r^2 &= -2y \\ x^2+y^2 &= -2y \\ x^2+y^2+2y &= 0 \\ x^2+y^2+2y+1 &= 1 \\ x^2+(y+1)^2 &= 1. \end{aligned}$$

This is an equation of a circle with center $(0,-1)$ and radius 1, so it has area $\pi\cdot 1^2 = \pi$.

22. Let M be the event "quiz on Monday", W be the event "quiz on Wednesday", and F be the event "quiz on Friday". If A is a set, I'll let A' denote its complement. Then

$$\begin{aligned} P(\text{exactly one quiz}) &= P(M\cap W'\cap F') + P(M'\cap W\cap F') + P(M'\cap W'\cap F) \\ &= 0.2\cdot 0.7\cdot 0.4 + 0.8\cdot 0.3\cdot 0.4 + 0.8\cdot 0.7\cdot 0.6 \\ &= 0.056 + 0.096 + 0.336 \\ &= 0.488. \end{aligned}$$

23. We can use the law of cosines which requires computing $\cos\theta$. Since $\theta = \tan^{-1}\frac{4}{3}$, $\tan\theta = \frac{4}{3}$. Since $0 < \theta < \frac{\pi}{2}$, we can use the following triangle to find $\cos\theta$:



Hence $\cos\theta = \frac{3}{5}$. Now let c denote the length of side AB. By the law of cosines,

$$\begin{aligned} c^2 &= 10^2 + 21^2 - 2\cdot 10\cdot 21\cos\theta \\ &= 100 + 441 - 420\cdot\frac{3}{5} \\ &= 289 \;\Rightarrow\; c = 17. \end{aligned}$$

24. Let x denote the height of the rectangle and y denote the length of the rectangle. Then $8-x$ is the height of the small triangle on top of the rectangle. This small triangle is similar to the large triangle and has a base of length y. Since corresponding parts of similar triangles are proportional, we obtain

$$\frac{y}{10}=\frac{8-x}{8} \quad\Rightarrow\quad y=\frac{10}{8}(8-x) \quad\Rightarrow\quad y=10-\frac{5}{4}x.$$

The area of the rectangle can now be expressed entirely in terms of x:

$$\begin{aligned} A &= xy \\ &= 10x-\frac{5}{4}x^2 \\ &= -\frac{5}{4}(x^2-8x) \\ &= -\frac{5}{4}(x^2-8x+16)+\frac{5}{4}\cdot 16 \\ &= -\frac{5}{4}(x-4)^2+20. \end{aligned}$$

The area is a maximum when $x=4$. The largest area is 20 in^2.

25. The inequality $|x^2-13x|<30$ is equivalent to

$$\begin{aligned} x^2-13x<30 \quad &\text{AND} \quad x^2-13x>-30 \\ x^2-13x-30<0 \quad &\text{AND} \quad x^2-13x+30>0 \\ (x-15)(x+2)<0 \quad &\text{AND} \quad (x-10)(x-3)>0 \end{aligned}$$

The solution set of the first inequality above is the interval $(-2,15)$ and the solution set of the second inequality above is $(-\infty,3)\cup(10,\infty)$. So the solution set of the inequality $|x^2-13x|<30$ is found by

$$(-2,15)\cap\big((-\infty,3)\cup(10,\infty)\big)=(-2,3)\cup(10,15).$$

26.
$$|x+2|-2|x+1|+|x|=\begin{cases} -(x+2)+2(x+1)-x & \text{if } x\le -2, \\ x+2+2(x+1)-x & \text{if } -2<x\le -1, \\ x+2-2(x+1)-x & \text{if } -1<x\le 0, \\ x+2-2(x+1)+x & \text{if } x>0. \end{cases}$$

$$=\begin{cases} 0 & \text{if } x\le -2, \\ 2x+4 & \text{if } -2<x\le -1, \\ -2x & \text{if } -1<x\le 0, \\ 0 & \text{if } x>0. \end{cases}$$

27. Substituting $x = y^2$ into the second equation gives

$$\begin{aligned}(y^2-3)^2 &= -8\left(y-\frac{3}{2}\right)\\ y^4-6y^2+9 &= -8y+12\\ y^4-6y^2+8y-3 &= 0.\end{aligned}$$

By inspection, $y = 1$ is a root of this equation, so $y-1$ is a factor. We can use synthetic division to obtain the other factor. We then have

$$(y-1)(y^3+y^2-5y+3) = 0.$$

By inspection, we see that $y = 1$ is a zero of the second factor, so we obtain another factor of $y-1$. Use synthetic division once more to obtain the quadratic factor. Now we have

$$\begin{aligned}(y-1)^2(y^2+2y-3) &= 0\\ (y-1)^2(y-1)(y+3) &= 0\\ (y-1)^3(y+3) &= 0.\end{aligned}$$

So $y = 1$ or $y = -3$. Now use the first equation: If $y = 1$ then $x = 1$. If $y = -3$ then $x = 9$. Therefore there are exactly 2 points of intersection of these two parabolas: $(1,1)$ and $(9,-3)$.

28. Let (r,s) be the point obtained by reflecting the point (a,b) about the line $y = -2x$. The slope of the line segment with endpoints (a,b) and (r,s) is $\frac{s-b}{r-a}$. Since this line segment is perpendicular to the line $y = -2x$, it must have slope $\frac{1}{2}$. This gives the equation

$$\begin{aligned}\frac{s-b}{r-a} &= \frac{1}{2}\\ s-b &= \frac{1}{2}r-\frac{1}{2}a\\ -\frac{1}{2}r+s &= -\frac{1}{2}a+b\\ r-2s &= a-2b.\end{aligned}$$

The midpoint of the line segment with endpoints (a,b) and (r,s) is $(\frac{a+r}{2}, \frac{b+s}{2})$. This point must lie on the line $y = -2x$, which gives the equation

$$\begin{aligned}\frac{b+s}{2} &= -2\left(\frac{a+r}{2}\right)\\ b+s &= -2a-2r\\ 2r+s &= -2a-b.\end{aligned}$$

We now have a system of two equations in the unknowns r and s:

$$\begin{aligned} r - 2s &= a - 2b, \\ 2r + s &= -2a - b. \end{aligned}$$

Adding the first equation to twice the second gives

$$5r = a - 2b - 4a - 2b \quad \Rightarrow \quad 5r = -3a - 4b \quad \Rightarrow \quad r = -\frac{3}{5}\,a - \frac{4}{5}\,b.$$

If we take -2 times the first equation and add it to the second we obtain

$$5s = -2a + 4b - 2a - b \quad \Rightarrow \quad 5s = -4a + 3b \quad \Rightarrow \quad s = -\frac{4}{5}\,a + \frac{3}{5}\,b.$$

Hence the point (r, s) is given by $\left(-\frac{3}{5}\,a - \frac{4}{5}\,b,\, -\frac{4}{5}\,a + \frac{3}{5}\,b\right)$.

29. **Solution 1**: Note that

$$\begin{aligned} x^3 &= (\,(x+2) - 2)^3 \\ &= (x+2)^3 - 3 \cdot 2(x+2)^2 + 3 \cdot 2^2(x+2) - 2^3 \\ &= (x+2)^3 - 6(x+2)^2 + 12(x+2) - 8, \end{aligned}$$

so $c = -6$.

Solution 2: We can expand the right hand side and equate the coefficients of the powers of x:

$$\begin{aligned} x^3 &= a + b(x+2) + c(x+2)^2 + d(x+2)^3 \\ &= a + bx + 2b + c(x^2 + 4x + 4) + d(x^3 + 6x^2 + 12x + 8) \\ &= a + 2b + 4c + 8d + (b + 4c + 12d)x + (c + 6d)x^2 + dx^3 \end{aligned}$$

Equating powers of x gives the following system:

$$\begin{aligned} a + 2b + 4c + 8d &= 0, \\ b + 4c + 12d &= 0, \\ c + 6d &= 0, \\ d &= 1. \end{aligned}$$

Substituting $d = 1$ into the third equation gives $c = -6$.

Solution 3: Let $f(x) = x^3$. The Taylor series for $f(x)$ about -2 is given by

$$f(x) = f(-2) + f'(-2)(x+2) + \frac{f''(-2)}{2!}\,(x+2)^2 + \frac{f'''(-2)}{3!}\,(x+2)^3 + \cdots.$$

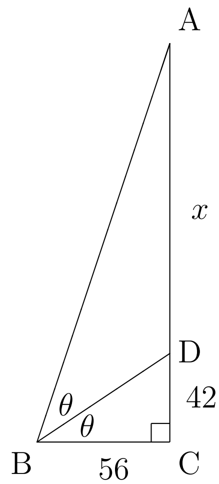
In this case, $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for all $n \geq 4$. So observe that the above Taylor series has a finite number of nonzero terms. We compute

$$f(-2) = -8,\ f'(-2) = 12,\ f''(-2) = -12,\ f'''(-2) = 6.$$

Substituting these values into the above Taylor series gives

$$x^3 = -8 + 12(x+2) - 6(x+2)^2 + (x+2)^3 \ \Rightarrow\ c = -6.$$

30. Let θ denote the measure of $\angle ABD$ which is also the measure of $\angle DBC$. Let x be the length of AD.



Using right triangle $\triangle BCD$ we see that $\tan\theta = \frac{42}{56} = \frac{3}{4}$. Using right triangle $\triangle BCA$ we see that $\tan 2\theta = \frac{x+42}{56}$. Now make use of the double angle formula for tangent:

$$\begin{aligned}
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta} \\
\frac{x+42}{56} &= \frac{2\cdot\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} \\
\frac{x+42}{56} &= \frac{\frac{3}{2}}{\frac{7}{16}} \\
\frac{x+42}{56} &= \frac{24}{7} \\
7x+294 &= 1344 \\
x &= 150.
\end{aligned}$$

31. First of all

$$\begin{aligned}
w^2 &= \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)^2 \\
&= \frac{1}{4}+2\cdot\left(-\frac{1}{2}\right)\cdot\frac{\sqrt{3}}{2}\,i-\frac{3}{4} \\
&= -\frac{1}{2}-\frac{\sqrt{3}}{2}\,i,
\end{aligned}$$

$$
\begin{aligned}
w^3 &= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) \\
&= \left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \\
&= \frac{1}{4} + \frac{3}{4} = 1.
\end{aligned}
$$

Note that you could have also computed these powers of w by using De Moivre's formula since $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Now observe that

$$
1 + w + w^2 = 1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = 0.
$$

So we have

$$
\begin{aligned}
1 + w + w^2 + w^3 + \cdots + w^{2019} &= (1 + w + w^2) + w^3(1 + w + w^2) + w^6(1 + w + w^2) \\
&\qquad + \cdots + w^{2016}(1 + w + w^2) + w^{2019} \\
&= w^{2019} \\
&= (w^3)^{673} \\
&= 1.
\end{aligned}
$$

32. Let $U =$ the set of all positive integers less than or equal to 2018 and let A, B, C be the following subsets:

$$
\begin{aligned}
A &= \{x \in U \,|\, x \text{ is divisible by } 5\}, \\
B &= \{x \in U \,|\, x \text{ is divisible by } 7\}, \\
C &= \{x \in U \,|\, x \text{ is divisible by } 17\}.
\end{aligned}
$$

If S is a finite set then I'll let $|S|$ denote the number of elements of S. Using integer division with remainder,

$$
2018 = 403 \cdot 5 + 3 \;\Rightarrow\; |A| = 403.
$$

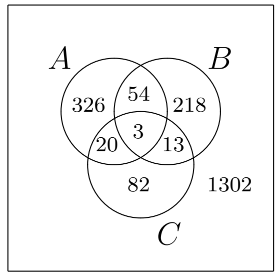
(Equivalently, $|A| = \left\lfloor \frac{2018}{5} \right\rfloor$ where $\lfloor x \rfloor =$ greatest integer $\leq x$.) Note that $|A \cap B|$ is the number of positive integers less than or equal to 2018 that are divisible by 35. So dividing:

$$
2018 = 57 \cdot 35 + 23 \;\Rightarrow\; |A \cap B| = 57.
$$

Continuing in this manner, we obtain:

$$
\begin{gathered}
|U| = 2018, \quad |A| = 403, \quad |B| = 288, \quad |C| = 118, \\
|A \cap B| = 57, \quad |A \cap C| = 23, \quad |B \cap C| = 16, \quad |A \cap B \cap C| = 3.
\end{gathered}
$$

The question is asking for $|(A \cup B) \cap C'|$. It is probably easiest to find this by filling in a Venn diagram using the above information:



We have $|(A \cup B) \cap C'| = 326 + 54 + 218 = 598.$